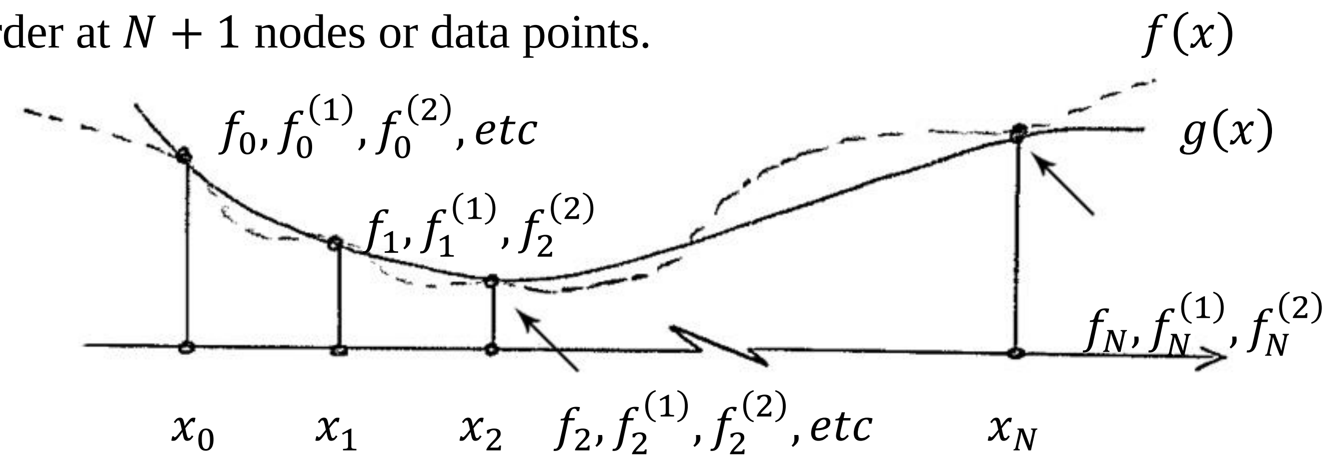
Lecture No. 8

Hermite Interpolation

- Develop an interpolating function $g(x)$ which equals the function and its derivatives up to p^{th} order at $N+1$ nodes or data points.



- Thus we require that

$g(x_i) = f_i \qquad i = 0, N \;\rightarrow N+1$ constraints

$g^1(x_i) = f_i^{(1)} \qquad i = 0, N$

$g^p(x_i) = f_i^{(p)} \qquad i = 0, N$

$\therefore (p+1)(N+1)$ constraints $\Rightarrow$

$g(x)$ will be a polynomial of degree $(p+1)(N+1) - 1$ (# of constraints must be equal # of unknowns coef.'s in $g(x)$!)

$$\therefore g(x) = \sum_{i=0}^{(N+1)(p+1)-1} a_i x^i$$

Example: Develops a two data point Hermite interpolation function which passes through the function and its first derivative

	x_i	f_i	$f_i^{(1)}$
x_0	0	f_0	$f_0^{(1)}$
x_1	+1	f_1	$f_1^{(1)}$

- 4 constraints $\Rightarrow g(x)$ is 3rd degree polynomial

  Note that $p = 1$ and $N + 1 = 2 \Rightarrow (p + 1)(N + 1) - 1 = 3$
- $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

  $g^{(1)}(x) = a_1 + 2a_2 x + 3a_3 x^2$
- Apply constraints

  $g(0) = f_0 \Rightarrow$

  $a_0 = f_0$

$$g(1) = f_1 \Rightarrow$$
$$a_0 + a_1 + a_2 + a_3 = f_1$$

$$g^{(1)}(0) = f_0^{(1)} \Rightarrow$$
$$a_1 = f_0^{(1)}$$

$$g^{(1)}(1) = f_1^{(1)} \Rightarrow$$
$$a_1 + 2a_2 + 3a_3 = f_1^{(1)}$$

- Write constraints eqs in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_1 \\ f_0^{(1)} \\ f_1^{(1)} \end{bmatrix}$$

$$\Rightarrow$$
$$a_0 = f_0$$
$$a_1 = f_0^{(1)}$$
$$a_2 = 3f_1 - 3f_0 - f_1^{(1)} - 2f_0^{(1)}$$
$$a_3 = -2f_1 + 2f_0 + f_0^{(1)} + f_1^{(1)}$$

$$\therefore \boxed{g(x) = f_0 + f_0^{(1)} x + \left(3f_1 - 3f_0 - f_1^{(1)} - 2f_0^{(1)}\right) x^2 + \left(-2f_1 + 2f_0 + f_0^{(1)} + f_1^{(1)}\right) x^3}$$

We note that constraints are indeed satisfied

$$g(0) = f_0 \quad g(1) = f_1 \quad g^{(1)}(0) = f_0^{(1)} \quad g^{(1)}(1) = f_1^{(1)}$$

- Re-writing (collecting terms $f_0; f_1$ etc)

$$g(x) = f_0(2x^3 - 3x^2 + 1) + f_1(-2x^3 + 3x^2) + f_0^{(1)}(x^3 - 2x^2 + x)$$
$$+ f_1^{(1)}(x^3 - 2x^2 + x) + f_1^{(1)}(x^3 - x^2)$$

$\Rightarrow$

$$\boxed{g(x) = f_0\phi_0(x) + f_1\phi_1(x) + f_0^{(1)}\psi_0(x) + f_1^{(1)}\psi_1(x)}$$

where $\phi_0(x) = 2x^3 - 3x^2 + 1$ associated with function value at node x_0

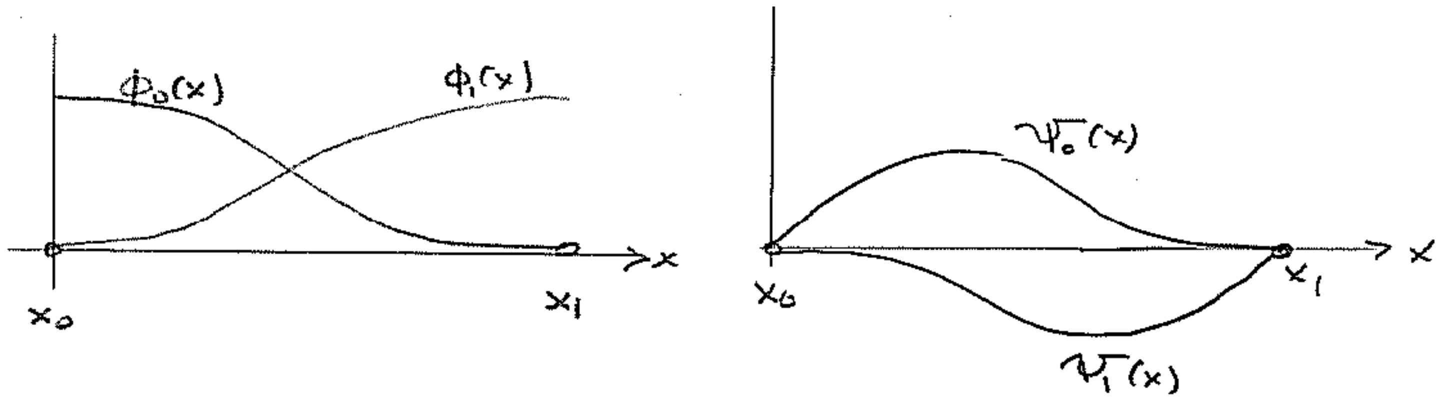
$\phi_1(x) = -2x^3 - 3x^2$ associated with function value at node x_1

$\psi_0(x) = x^3 - 2x^2 + x$ associated with first derivative value at node x_0

$\psi_1(x) = x^3 - x^2$ associated with first derivative value at node x_1

(note ϕ_0, ϕ_1 <u>different</u> from Lagrange interpolation basis functions)

∴ Each node has 2 interpolating basis function associated with it, one associated with the function value and one with first derivative. Each of the functions (2 func/node and 2 nodes) are cubics



- From our eq. for $g(x)$ we note that each interpolating basis function can be defined separately.
  - Each function is a cubic

– Constraint equations fall out

$$
\begin{array}{lcll}
g(x_0) = f_0 & \Rightarrow & \phi_0(x_0) = 1 & \phi_1(x_0) = 0 \\
g(x_1) = f_1 & \Rightarrow & \phi_0(x_1) = 0 & \phi_1(x_1) = 1 \\
g^{(1)}(x_0) = f_0^{(1)} & \Rightarrow & \phi_0^{(1)}(x_0) = 0 & \phi_1^{(1)}(x_0) = 0 \\
g^{(1)}(x_1) = f_1^{(1)} & \Rightarrow & \phi_0^{(1)}(x_1) = 0 & \phi_1^{(1)}(x_1) = 0
\end{array}
$$

$$\underbrace{\phantom{\phi_0^{(1)}(x_1) = 0 \quad \phi_1^{(1)}(x_1) = 0}}_{\substack{\phi_i(x_j)=\delta_{ij} \\ \phi_i^{(1)}(x_j)=0}}$$

$$
\underbrace{
\begin{array}{ll}
\psi_0(x_0) = 0 & \psi_1(x_0) = 0 \\
\psi_0(x_1) = 0 & \psi_1(x_1) = 0 \\
\psi_0^{(1)}(x_0) = 1 & \psi_1^{(1)}(x_0) = 0 \\
\psi_0^{(1)}(x_1) = 0 & \psi_1^{(1)}(x_1) = 1
\end{array}
}_{\substack{\psi_i(x_j)=0 \\ \psi_i^{(1)}(x_j)=\delta_{ij}}}
$$

16 constraints

Each interpolating basis function is defined as a cubic

$$\therefore \quad \phi_i(x) = a_i + b_i x + c_i x^2 + d_i x^3 \quad i = 0,1$$

$$\psi_i(x) = e_i + f_i x + g_i x^2 + h_i x^3 \quad i = 0,1$$

16 unknowns

- <u>General Hermite Interpolation</u>

  N data points/nodes j function, 1st derivative → pth derivative

$$g(x) = \sum_{i=0}^{N} \phi_i\,(x) f_i + \sum_{i=0}^{N} \psi_i(x) f_i^{(1)} + \cdots + \sum_{i=0}^{N} \Theta_i(x) f_i^{(P)}$$

- To satisfy

$$g(x_j) = f_j \Rightarrow \quad \sum_{i=0}^{N} \phi_i(x_j) f_i + \sum_{i=0}^{N} \psi_i(x_j) f_1^{(1)} + \cdots + \sum_{i=0}^{N} \Theta_i(x_j) f_i^{(P)} = f_j$$

  Requires the following constraints:

$$\phi_i(x_j) = \delta_{ij} \qquad i,j = 0,N$$

$$\psi_i(x_j) = 0 \qquad i,j = 0,N$$

$$\vdots \qquad\qquad \vdots$$

$$\Theta_i(x_j) = 0 \qquad i,j = 0,N$$

- To satisfy

$$g^{(1)}(x_j) = f_j^{(1)} \Rightarrow \sum_{i=0}^{N} \phi_i^{(1)}(x_j) f_i + \sum_{i=0}^{N} \psi_i^{(1)}(x_j) f_i^{(1)} + \cdots + \sum_{i=0}^{N} \Theta_i^{(1)}(x_j) f_i^{(P)} = f_j^{(1)}$$

Leads to 2nd set of $(p+1)(N+1)$ constraints

$$\phi_i^{(1)}(x_j) = 0 \qquad i, j = 0, N$$

$$\psi_i^{(1)}(x_j) = \delta_{ij} \qquad i, j = 0, N$$

$$\vdots \qquad i, j = 0, N$$

$$\Theta_i^{(1)}(x_j) = 0 \qquad i, j = 0, N$$

- $(P+1)^{th}$ set of $(p+1)(N+1)$ constraints:

$$\phi_i^{(P)}(x_j) = 0 \qquad i, j = 0, N$$

$$\psi_i^{(P)}(x_j) = 0 \qquad i, j = 0, N$$

$$\vdots \qquad i, j = 0, N$$

$$\Theta_i^{(P)}(x_j) = \delta_{ij} \qquad i, j = 0, N$$

- Each set of (interpolating) basis functions has the general form

$$\phi_i(x) = \sum_{j=0}^{(p+1)(N+1)-1} a_{i,j} x^j \qquad i = 0, N$$

$$\psi_i(x) = \sum_{j=0}^{(p+1)(N+1)-1} b_{i,j} x^j \qquad i = 0, N$$

$$\vdots$$

$$\Theta_i(x) = \sum_{j=0}^{(p+1)(N+1)-1} t_{i,j} x^j \qquad i = 0, N$$

Applying Hermite Interpolation to Develop u_{app}

- Develop an approximation u_{app} which is the sum of Localized Hermite approximations which satisfy C_1 functional continuity

$$u_{app} = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \left[u_i^j \phi_i^j + u_i^{(1)^j} \psi_i^j \right]$$

where u_i^j = functional value at node *i* within element *j*

ϕ_i^j = Hermite basis function at node *i* within element *j* associated with the function value. All ϕ_i^j are equal to zero outside of element *j*

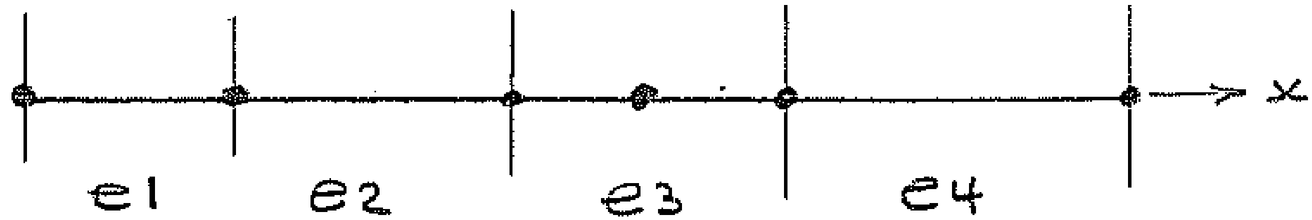
Also note that these are <u>not</u> the same function as were used in Lagrange interpolation

$u_i^{(1)^j}$ = first derivative value at node *i* within element *j*

ψ_i^j = Hermite basis function at node *i* within element *j* associated with the first derivative value. All Ψ_i^j are equal to zero outside of element *j*

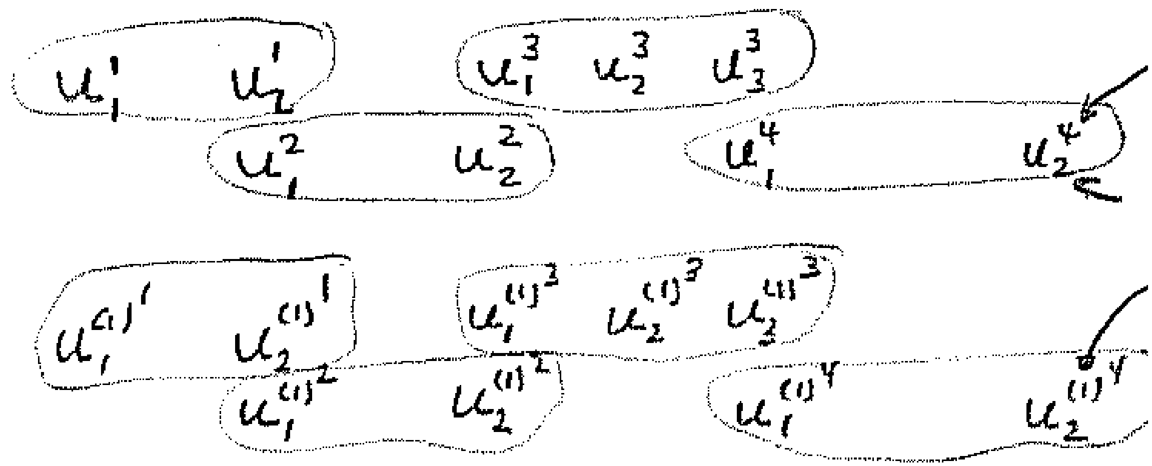
$j = 1, M$ = total no. of finite elements

$i = 1, N_j$ = total no. of nodes within element *j* element



local unknown function values

local unknown derivative values



element index

node index

denotes derivative

- Thus the unknown expansion coefficients are now equal to the function and first derivative values at the nodes.
- To ensure C_1 inter-element functional continuity we must have at all inter-element boundaries:

  u_{app} (to the left of an inter-element boundary) $=$ u_{app} (to the right of an inter-element boundary)

- For the example case given:
  - Anywhere in element 1

$$u^1(x) = u_1^1\phi_1^1(x) + u_2^1\phi_2^1(x) + u_1^{(1)^1}\psi_1^1(x) + u_2^{(1)^1}\psi_2^1(x)$$

  Note that all other Hermite basis functions from other elements are defined as zero

  - Anywhere in element 2

$$u^2(x) = u_1^2\phi_1^2(x) + u_2^2\phi_2^2(x) + u_1^{(1)^2}\psi_1^2(x) + u_2^{(1)^2}\psi_2^2(x)$$

  - Element 1 evaluated at r.h.s.

  $u^1(x_2^1) = u_2^1$ (since only $\phi_2^1(x_2^1) = 1$ and all other basis functions equal zero at x_2^1)

  - Element 2 evaluated at l.h.s

  $u^2(x_1^2) = u_1^2$ (since only $\phi_1^2(x_1^2) = 1$ and all other basis functions equal zero at x_1^2)

  - To ensure C_1 functional continuity we must have functional values equal at the inter-element boundaries. Thus we must have functional expansion coef.'s equal at inter-element boundaries.

$$u_2^1 = u_1^2$$

  - Other functional unknowns are related as (for our example)

$$u_3^2 = u_1^3$$

$$u_3^3 = u_1^4$$

- To ensure C_1 inter-element functional continuity we must <u>also</u> have at all inter-element boundaries

  $\frac{du_{app}}{dx}$ (to the left of an inter-element boundary) $=\frac{du_{app}}{dx}$ (to the right of an inter-element boundary)

- For the example case give

  – Anywhere in element 1

$$\frac{du^1(x)}{dx}=u_1^1\frac{d\phi_1^1(x)}{dx}+u_2^1\frac{d\phi_2^1(x)}{dx}+u_1^{(1)^1}\frac{d\psi_1^1(x)}{dx}+u_2^{(1)^1}\frac{d\psi_2^1(x)}{dx}$$

  Note that all other Hermite basis functions from other elements are defined as zero

  – Anywhere in element 2

$$\frac{du^2(x)}{dx}=u_1^2\frac{d\phi_1^2(x)}{dx}+u_2^2\frac{d\phi_2^2(x)}{dx}+u_1^{(1)^2}\frac{d\psi_1^2(x)}{dx}+u_2^{(1)^2}\frac{d\psi_2^2(x)}{dx}$$

  – Evaluate derivative for element 1 at r.h.s.

$$\frac{du^1(x_2^1)}{dx}=u_2^{(1)^1}$$

  (since only $\frac{d\psi_2^1(x_2^1)}{dx}=1$ and all other derivatives of basis functions evaluated at x_2^1 are equal to zero)

– Evaluate derivative for element 2 at l.h.s.

$$\frac{du^2(x_1^2)}{dx} = u_1^{(1)^2}$$

(Since only $\frac{d\psi_1^2(x_1^2)}{dx} = 1$ and all other derivatives of basis functions evaluated at x_1^2 are equal to zero)

– To ensure c_1 functional continuity we must have first derivative values equal at inter-element boundaries. Thus we must have first derivative expansion coef.'s equal at inter-element boundaries:

$$u_2^{(1)^1} = u_1^{(1)^2}$$

– Other first derivative unknowns are related as (for our example)

$$u_2^{(1)^2} = u_1^{(1)^3}$$

$$u_3^{(1)^3} = u_1^{(1)^4}$$

- Again there are several approaches to implement the inter-element functional and first derivative continuity (i.e. setting equal the adjacent inter-element function value and first derivative expansion coefficients)

- Develop “Cardinal” basis functions which are formed by patching together the various localized Hermite basis function and defining them globally. This also implies that you redefine the expansion coef.’s globally (so that now there will only be 2 coef.’s per node, one function value and one first derivative value)
- Actually implement all expansions locally. Then take care of inter-element functional continuity by assembling the “global” matrix correctly

- Now both function and first derivative b.c.’s can be easily incorporated for 1-D problems
  - Separately i.e. either function or first derivative
  - As a pair of “essential” b.c.’s associated with a fourth order operator
- Even in multiple dimensional problems, function and first derivatives specified b.c.’s can be easily incorporated.
- Note that b.c. implementation/satisfaction and inter-element functional continuity enforcement are made simple and possible by the use of the Hermite basis functions. We must however place nodes at the ends of the domain as well as at the ends of each element for this to work.
- The Hermite basis functions $\phi_i^j(x)$ and $\psi_i^j(x)$ are linearly independent

Example

Solve

$\frac{d^4u}{dx^4} = 0$ for $\in [x_L, x_R]$

Essential b.c. are specified as

$$u(x = x_L) = u_L$$

$$\left.\frac{du}{dx}\right|_{x=x_L} = u_L^{(1)}$$

$$u(x = x_R) = u_R$$

$$\left.\frac{du}{dx}\right|_{x=x_R} = u_R^{(1)}$$

Consider the following 4 global nodes defined over 3 elements

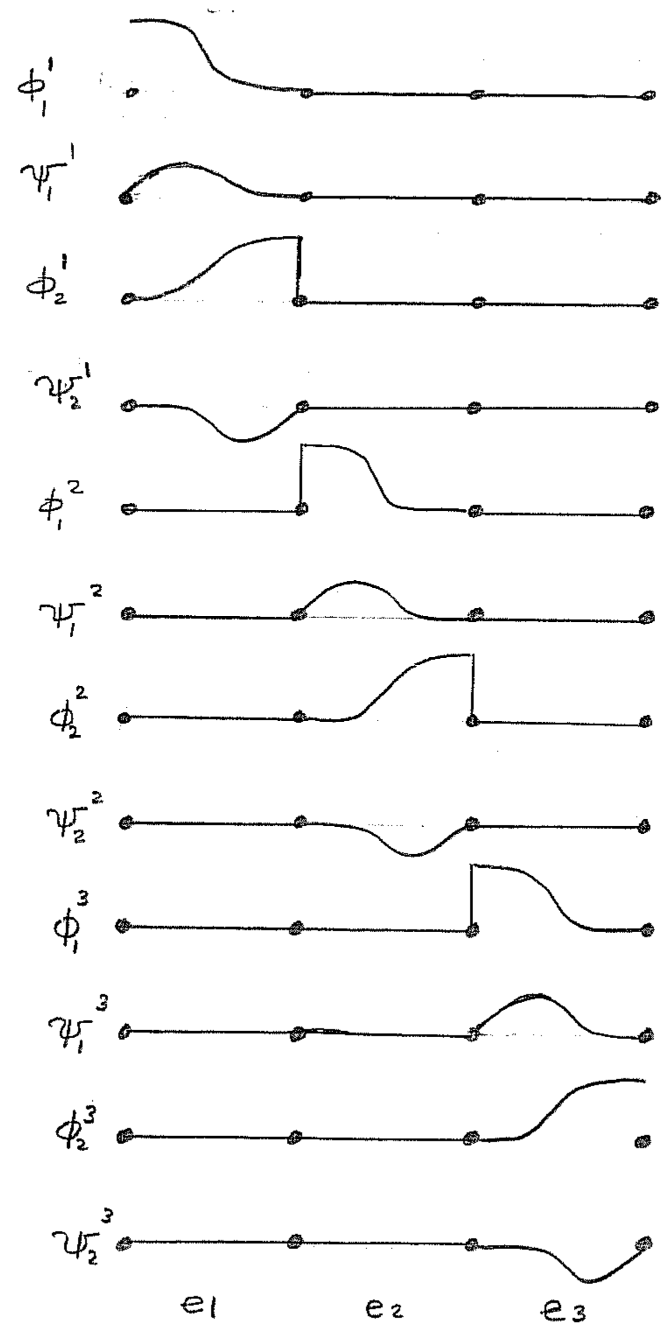
element	}	e1	}	e2	}	e3	}
global node no.	n1		n2		n3		n4
local node no.	1		2 1		2 1		2
global function	u_{n1}		u_{n2}		u_{n3}		u_{n4}
global derivative	$u^{(1)}_{n1}$		$u^{(1)}_{n2}$		$u^{(1)}_{n3}$		$u^{(1)}_{n4}$
local function	u_1^1		u_2^1 u_1^2		u_2^2 u_1^3		u_2^3
local derivative	$u_1^{(1)^1}$		$u_2^{(1)^1}$ $u_1^{(1)^2}$		$u_2^{(1)^2}$ $u_1^{(1)^3}$		$u_2^{(1)^3}$

- We will have the following elemental base expansion

$$u_{app} = \overbrace{u_L\phi_1^1 + u_L^{(1)}\psi_1^1}^{part\ of\ u_B} + u_2^1\phi_2^1 + u_2^{(1)^1}\psi_2^1 + u_1^2\phi_1^2 + u_1^{(1)^2}\psi_1^2$$

$$+ u_2^2\phi_2^2 + u_2^{(1)^2}\psi_2^2 + u_1^3\phi_1^3 + u_1^{(1)^3}\psi_1^3$$

$$+ \underbrace{u_R\phi_2^3 + u_R^{(1)}\psi_2^3}_{part\ of\ u_B}$$



- This expansion has 8 local unknowns

- 2 functional and 2 derivative inter-element constraints will be enforced to ensure C_1 continuity

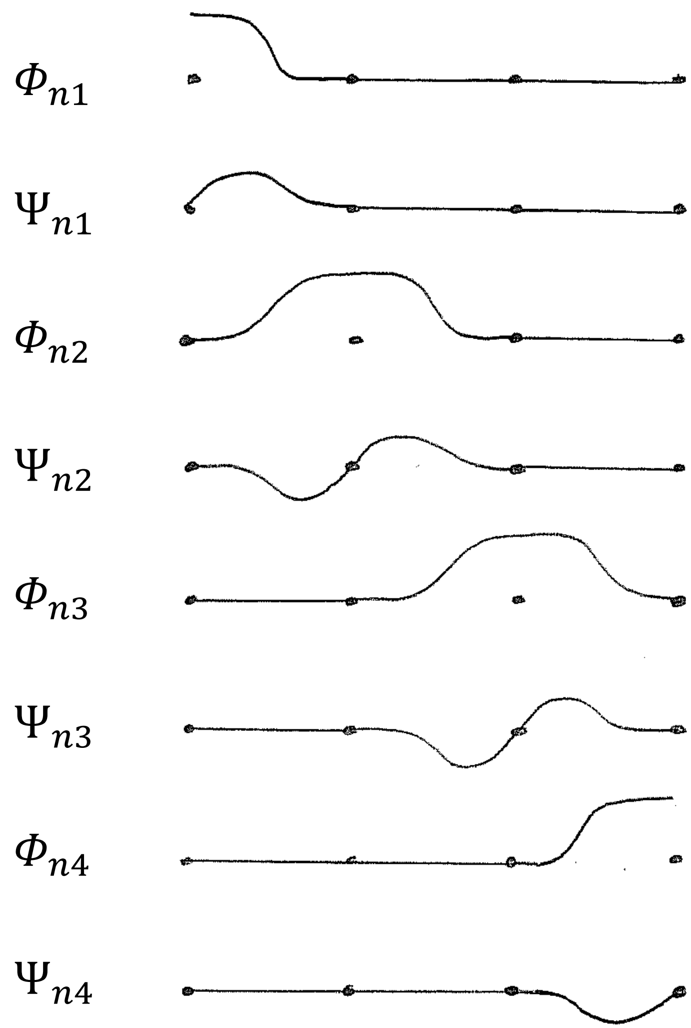
- We can also patch together these functions and form "cardinal" bases.

  This will lead to the following global expansion.

$$u_{app} = u_L \Phi_{n1} + u_L^{(1)} \Psi_{n1} + u_{n2} \Phi_{n2} + u_{n2}^{(1)} \Psi_{n2} + u_{n3} \Phi_{n3} + u_{n3}^{(1)} \Psi_{n3} + u_R \Phi_{n4} + u_R^{(1)} \Psi_{n4}$$

$\Phi_{n1} = \phi_1^1$	$u_{n1} = u_1^1 = u_L$	essential b.c.
$\Psi_{n1} = \psi_1^1$	$u_{n1}^{(1)} = u_1^{(1)^1} = u_L^{(1)}$	essential b.c.
$\Phi_{n2} = \phi_2^1 + \phi_2^1$	$u_{n2} = u_2^1 = u_1^2$	unknown
$\Psi_{n2} = \psi_2^1 + \psi_1^2$	$u_{n2}^{(1)} = u_2^{(1)^1} = u_1^{(1)^2}$	unknown
$\Phi_{n3} = \phi_2^2 + \phi_1^3$	$u_{n3} = u_2^2 = u_1^3$	unknown
$\Psi_{n3} = \psi_2^2 + \psi_1^3$	$u_{n3}^{(1)} = u_2^{(1)^2} = u_1^{(1)^3}$	unknown
$\Psi_{n4} = \psi_2^3$	$u_{n4} = u_2^3 = u_R$	essential b.c.
$\Psi_{n4} = \psi_2^3$	$u_{n4}^{(1)} = u_2^{(1)^3} = u_R^{(1)}$	essential b.c.
Cardinal or Global functions / local functions	global unknown function and derivatives / local unknown functions and derivatives	

- Note that the global or cardinal basis functions and the associated expansion u_{app} automatically satisfy the C_1 functional continuity. This is not true for the local expansions for which we must still enforce the functional continuity constraints. However this will be easy to do! Overall working with local expansions will be much easier!



Hermite Cubic Basis Functions

- Form an approximation which has:
  - Functional continuity
  - Continuity of the first derivative
  - Piecewise continuous 2nd derivatives
- Advantages
  - Gives highly accurate interpolation functions.
  - Have enough functional continuity to be used with FE collocation (for $L(u)$ 2nd order) or for any formulation that requires C_1 or higher continuity (e.g. symmetrical weak form of a 4th order p.d.e)

Construction of Hermite Cubics

Desire both inter-element continuity of the function and the first derivative.

Thus p and $p_{,x}$ are the unknowns at 2 end nodes.

We need 4 parameters and 2 nodes per element.

The simplest polynomial for the element is:

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

Number of unknowns	4N
Number of continuity Constraints	2(N-1)
Number of global unknowns	2(N+1)

At each node we will define 2 basis functions $\phi^{(i)}(x)$ and $\psi^{i}(x)$, one associated with the function and the second associated with the derivative

Deviation of Hermite Cubic Basis Functions for the unit element

$$\phi_i(\xi) = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3, \quad i = 1,2$$

and

$$\psi_i(\xi) = e_i + f_i\xi + g_i\xi^2 + h_i\xi^3, \quad i = 1,2$$

thus we have a total of 16 unknown coefficients.

- We define 8 constraint equations by requiring that:

  $\phi_i(\xi_j) = \delta_{ij}$ and $\frac{d\phi_i}{d\xi}(\xi_j) = 0, \quad i,j = 1,2$

  Thus ϕ has values of 1 and 0 and 0 and 1 for the 2 nodes and always has zero first derivatives at the 2 nodes.

- Furthermore we define an additional 8 constraint equations by requiring that:

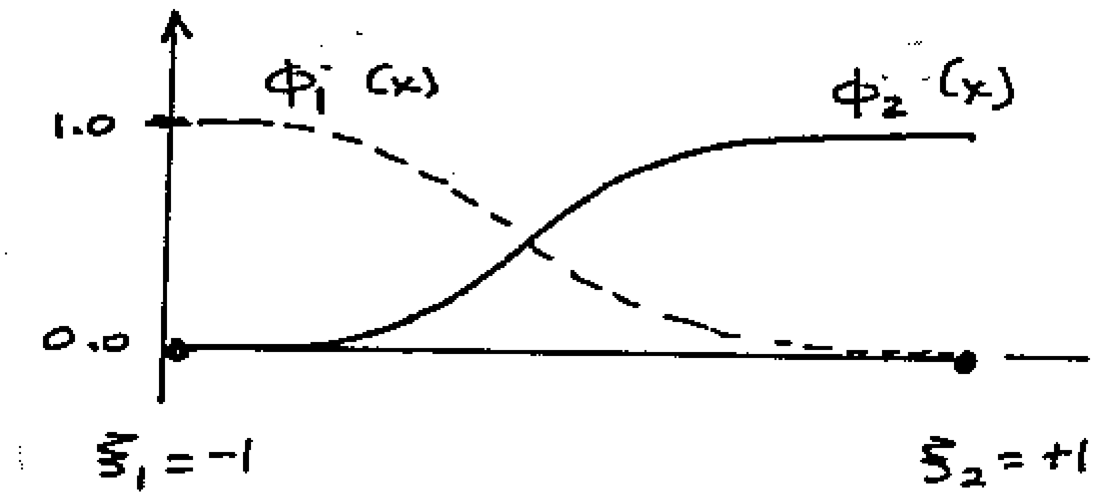
  $\psi_i(\xi_j) = 0$ and $\frac{d\psi_i}{d\xi}(\xi_j) = \delta_{ij}\frac{\Delta x}{2} \quad i,j = 1,2$

  Hence ψ always has a value of 0 and $\frac{d\psi}{d\xi}$ has values of ($\frac{\Delta x}{2}$ and 0) and (0 and $\frac{\Delta x}{2}$) for the 2 nodes.

- Thus we have a total of 16 unknowns and 16 constraint equations.
- Let the nodes for the unit element be $\xi_1 = -1$ and $\xi_2 = +1$. Applying the above constraints we obtain:

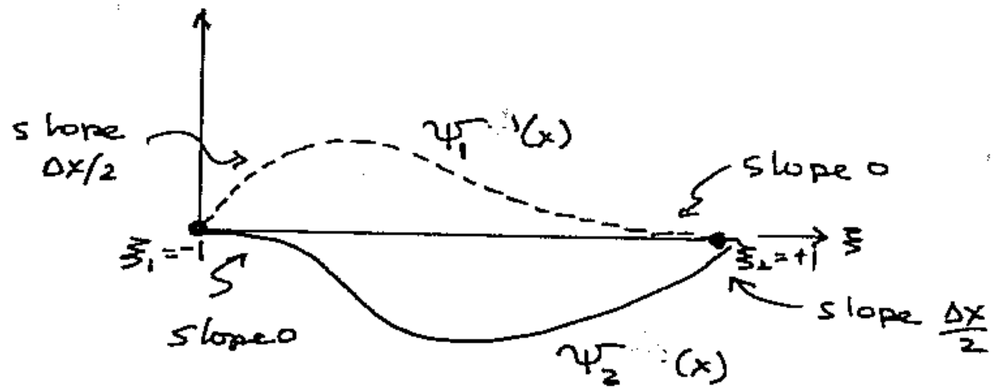
$$\phi_1(\xi) = \frac{1}{4}(\xi - 1)^2(\xi + 2)$$
$$\phi_2(\xi) = \frac{1}{4}(2 - \xi)(\xi + 1)^2$$
$$\psi_1(\xi) = \frac{\Delta x}{8}(\xi - 1)^2(\xi + 1)$$
$$\psi_2(\xi) = \frac{\Delta x}{8}(\xi - 1)(\xi + 1)^2$$

- ϕ_1 and ϕ_2 are plotted as:



Note that the slope is always zero at the nodes

- ψ_1 and ψ_2 are plotted as follows:



Note that these functions are always zero at the nodes:

- The approximating function over the element is defined as:

$$\hat{u}^e = u_1^e \phi_1^e + u_2^e \phi_2^e + u_1^{(1)^e} \psi_1^e + u_2^{(1)^e} \psi_2^e$$

- The unknown coefficients of ϕ at the nodes equal the functional values at the nodes:

  $\hat{u}_e^e(\xi_1) = u_1^e$ and $\hat{u}_e^e(\xi_2) = u_2^e$

- The unknown coefficients of ψ at the nodes are related to the slope at the nodes:

$$\frac{d\hat{u}^e}{d\xi}(\xi_1) = u_1^{(1)^e} \frac{\Delta x}{2}$$

$$\frac{d\hat{u}^e}{d\xi}(\xi_2) = u_2^{(1)^e} \frac{\Delta x}{2}$$

However

$$\left.\frac{d\hat{u}^e}{dx}\right|_{x=x_1} = \left.\frac{d\hat{u}^e}{d\xi}\right|_{\xi=\xi_1} \frac{d\xi}{dx} = u_1^{(1)^e} \frac{\Delta x}{2} \frac{2}{\Delta x} = u_1^{(1)^e}$$

similarly

$$\left.\frac{d\hat{u}^e}{dx}\right|_{x=x_2} = u_1^{(1)^e}$$

Therefore the unknown coefficients of* ψ *evaluated at the nodes equal the derivative of the function (globally) at the node.

- Basis functions are related to local basis functions by the same coordinate transformation as for the Lagrange basis functions:

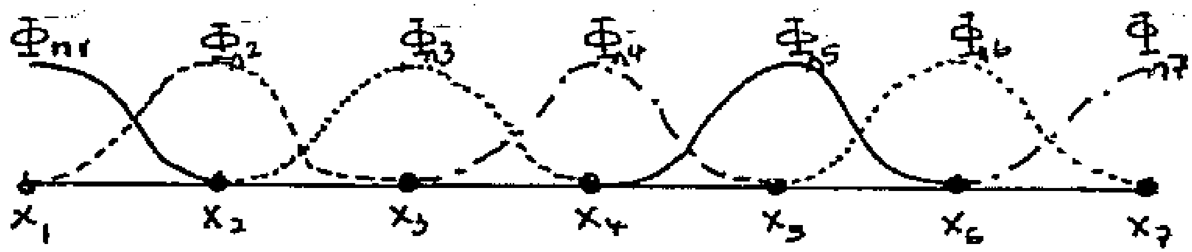
$$\phi_1(\xi) = \phi_{2j-1}\big(x(\xi)\big) = \phi_{2j-1}(x)$$

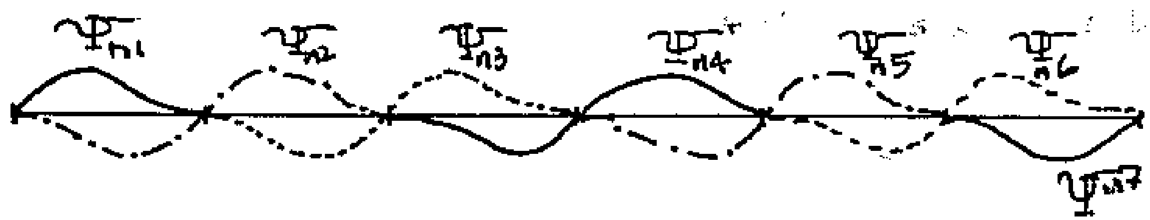
$$\phi_2(\xi) = \phi_{2j}\big(x(\xi)\big) = \phi_{2j}(x)$$

$$\psi_1(\xi) = \psi_{2j-1}\big(x(\xi)\big) = \psi_{2j-1}(x)$$

$$\psi_2(\xi) = \psi_{2j}\big(x(\xi)\big) = \psi_{2j}(x)$$

- Taking into account the functions we generated and their <u>associated continuity constraints</u> we have the following $2(N+1)$“Cardinal” Basis (these a priori incorporate functional continuity):





- Hence we have

$$\hat{u}(x) = \sum_{n_{i=1}}^{N+1} u_{ni}\Phi_{ni}(x) + \sum_{n_{i=1}}^{N+1} u_{ni}^{(1)}\Psi_{ni}$$

- We note that at node i:

u_{ni} = global functional value at node *ni*

$u_{ni}^{(1)}$ = global derivative calue at node *ni*

It's very useful and convenient to solve directly for both the function value and its first derivative at the nodes:

- Derivatives in general (not at the nodes) are evaluated as:

$$\frac{d\hat{u}}{dx} = \sum_{ni=1}^{N+1} u_{ni} \frac{d\Phi_{ni}}{dx} + \sum_{i=1}^{N+1} u_{ni}^{(1)} \frac{d\Psi_{ni}}{dx}$$

for local basis functions:

$$\frac{d\hat{u}^e}{dx} = u_1^e \frac{d\phi_1^e}{d\xi}\frac{d\xi}{dx} + u_2^e \frac{d\phi_2^e}{d\xi}\frac{d\xi}{dx} + u_1^{(1)^e} \frac{d\psi_1^e}{d\xi}\frac{d\xi}{dx} + u_2^{(1)^e} \frac{d\psi_2^e}{d\xi}\frac{d\xi}{dx}$$

We note that

$$\frac{d\xi}{dx} = \frac{\Delta x}{2}$$

Summary of Basis Functions

Basis	No. of Unknowns		Bandwidth of matrix produced[a]	D.O.F. per node[b]	Functional continuity
	primary	cardinal			
Linear Lagrange	2N	N+1	3	1	C_0
Quadratic lagrange	3N	2N+1	5	1	C_0
Cubic Lagrange	4N	3N+1	6	1	C_0
Cubic Hermite	4N	2N+2	7	2	C_1

a. For a 1-D problem numbered sequentially

b. Degrees of freedom